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THE EULER-HEISENBERG LAGRANGIAN BEYOND ONE LOOP

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We review what is presently known about higher loop corrections to the Euler-Heisenberg Lagrangian and its Scalar QED analogue. The use of those corrections as a tool for the study of the properties of the QED perturbation series is outlined. As a further step in a long-term effort to prove or disprove the convergence of the N photon amplitudes in the quenched approximation, we present a parameter integral representation of the three-loop Euler-Heisenberg Lagrangian in 1+1 dimensional QED, obtained in the worldline formalism.

Keywords: Euler-Heisenberg, Multiloop, Worldline Formalism.

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1. Introduction

Heisenberg and Euler's 1936 calculation [1] of the one-loop effective Lagrangian induced for a constant Maxwell field by a spinor loop was not only a milestone in the development of QED, but remains until today the prototypical example for the concept of integrating out degrees of freedom in field theory. In practical terms, it encodes in a very concise form information on a host of photonic low-energy processes (see 2 for a review). Higher loop corrections to the Euler-Heisenberg Lagrangian and its Scalar QED analogue, due to Weisskopf [3] (both called "EHL" in the following for simplicity) were studied only much later, starting with Ritus' 1975 calculation [4] of the two-loop EHL. The purpose of this talk is to give a summary

on what is known about the EHL at the multiloop level, and to argue that those multiloop corrections, although not likely to be of phenomenological interest in the near future, contain important structural information on QED.

2. QED in a constant external field in the worldline formalism

We start with a short introduction to the worldline representation of the QED S-matrix, going back to Feynman [5,6], which in recent years has emerged as an extremely efficient tool for the computation of processes involving constant external fields in QED. For the simplest case of the one-loop effective action in Scalar QED, it reads [5]

$$\Gamma[A] = \int_0^\infty \frac{dT}{T}\, \mathrm{e}^{-m^2T} \int \mathcal{D}x \exp\left[-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ieA_\mu \dot{x}^\mu\right)\right] \tag{2.1}$$

Here m and T are the mass and proper-time of the scalar particle in the loop, and the path integral runs over the space of closed trajectories with period T, $x^\mu(T) = x^\mu(0)$, in (euclidean) spacetime. The spinor QED equivalent of (2.1) is obtained [6] by the addition of a global factor of $-\frac{1}{2}$, and the insertion of a *spin factor* $S[x,A]$ under the path integral. The modern way of writing this spin factor is in terms of an additional Grassmann path integral [7],

$$S[x,A] = \int \mathcal{D}\psi(\tau)\, \exp\left[-\int_0^T d\tau \left(\frac{1}{2}\psi\cdot\dot{\psi} - ie\psi^\mu F_{\mu\nu}\psi^\nu\right)\right] \tag{2.2}$$

Here the path integration is over the space of anticommuting functions antiperiodic in proper-time, $\psi^\mu(\tau_1)\psi^\nu(\tau_2) = -\psi^\nu(\tau_2)\psi^\mu(\tau_1)$, $\psi^\mu(T) = -\psi^\mu(0)$.

Presently, three quite different methods are available for computing such path integrals (see 8 for a review): (i) the "string-inspired approach", based on a perturbative expansion and gaussian path integration [9,10,11,12,13] (ii) the "worldline instanton approach", using a stationary path approximation [14,15] and (iii) the direct numerical calculation using Monte Carlo techniques [16].

All three methods have been applied to the calculation of Euler-Heisenberg Lagrangians. We will explain here only the "string-inspired" method; see 14, 15, 17 for the worldline instanton and 16, 18 for the worldline Monte Carlo approach. If we expand the interaction exponential,

$$\exp\Big[-\int_0^T d\tau\, ieA_\mu\dot{x}^\mu\Big] = \sum_{N=0}^\infty \frac{(-ie)^N}{N!} \prod_{i=0}^N \int_0^T d\tau_i \Big[\dot{x}^\mu(\tau_i)A_\mu(x(\tau_i))\Big] \tag{2.3}$$

the individual terms correspond to Feynman diagrams describing a fixed number of interactions of the scalar loop with the external field, see fig. 1.

The corresponding $N-$photon scattering amplitude is then obtained by specializing to a background consisting of a sum of plane waves with definite polarizations,

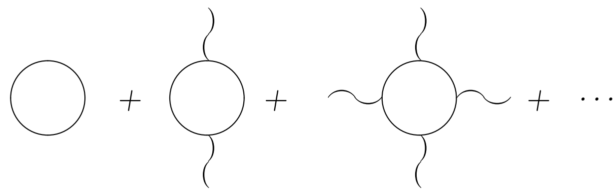

Fig. 1. Feynman diagrams equivalent to the one-loop effective action.

$A_\mu(x) = \sum_{i=1}^N \varepsilon_{i\mu} \mathrm{e}^{ik_i\cdot x}$, and picking out the term containing every ε_i once. This yields the following representation of the N - photon amplitude,

$$\Gamma[\{k_i,\varepsilon_i\}\}] = (-ie)^N \int_0^\infty \frac{dT}{T}\, e^{-m^2 T} \int \mathcal{D}x\, V^A_{\rm scal}[k_1,\varepsilon_1]\dots V^A_{\rm scal}[k_N,\varepsilon_N] \mathrm{e}^{-\int_0^T d\tau \frac{\dot{x}^2}{4}} \tag{2.4}$$

Here $V^A_{\rm scal}[k,\varepsilon] \equiv \int_0^T d\tau\, \varepsilon\cdot\dot{x}(\tau)\,\mathrm{e}^{ikx(\tau)}$ denotes the same photon vertex operator used in string perturbation theory. The integral over the zero mode $x_0^\mu = \frac{1}{T}\int_0^T d\tau\, x^\mu(\tau)$ factors out and produces the usual energy-momentum conservation factor. The reduced path integral $\int \mathcal{D}y(\tau)$ over $y(\tau) \equiv x(\tau) - x_0$ is gaussian, and can be evaluated using the "bosonic" worldline Green's function G_B,

$$\langle y^\mu(\tau_1) y^\nu(\tau_2)\rangle = -g^{\mu\nu} G_B(\tau_1,\tau_2) = -g^{\mu\nu}\Bigl[\mid \tau_1 - \tau_2 \mid - \frac{(\tau_1-\tau_2)^2}{T}\Bigr] \tag{2.5}$$

Using a formal exponentiation of the factors $\varepsilon_i \cdot \dot{x}_i$'s and "completing the square" yields the following closed-form expression for the one-loop N - photon amplitude:

$$\begin{aligned}\Gamma[\{k_i,\varepsilon_i\}] &= (-ie)^N (2\pi)^D \delta(\sum k_i) \int_0^\infty \frac{dT}{T} (4\pi T)^{-\frac{D}{2}} e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i \\ &\times \exp\biggl\{\sum_{i,j=1}^N \bigl[\frac{1}{2} G_{Bij} k_i\cdot k_j + i\dot{G}_{Bij} k_i \cdot \varepsilon_j + \frac{1}{2}\ddot{G}_{Bij}\varepsilon_i\cdot\varepsilon_j\bigr]\biggr\}\mid_{\rm multi-linear}\end{aligned} \tag{2.6}$$

Here it is understood that only the terms linear in all the $\varepsilon_1,\dots,\varepsilon_N$ have to be taken. Dots generally denote a derivative acting on the first variable, and we abbreviate $G_{Bij} \equiv G_B(\tau_i,\tau_j)$ etc. The factor $(4\pi T)^{-\frac{D}{2}}$ represents the free Gaussian path integral determinant. The expression (2.6) is identical with the "Bern-Kosower Master Formula" for the N photon case [10,11,13].

In the spinor QED case, the correlator for the evaluation of the additional Grassmann path integral is $\langle \psi(\tau_1)\psi(\tau_2)\rangle = \frac{1}{2} g^{\mu\nu} G_F(\tau_1,\tau_2)$, with $G_F(\tau_1,\tau_2) =$

$\mathrm{sign}(\tau_1 - \tau_2)$. Its explicit evaluation can, however, be circumvented, using the following "replacement rule" [10]: Writing out the exponential in the master formula eq.(2.6) for a fixed number N of photons, one obtains an integrand

$$\exp\Bigl\{\Bigr\}\,|_{\mathrm{multi-linear}} \quad = (-i)^N P_N(\dot{G}_{Bij}, \ddot{G}_{Bij}) \exp\biggl[\frac{1}{2}\sum_{i,j=1}^{N} G_{Bij} k_i \cdot k_j\biggr] \tag{2.7}$$

with a certain polynomial P_N depending on the various $\dot{G}_{Bij}$'s, $\ddot{G}_{Bij}$'s, as well as on the kinematic invariants. By suitable partial integrations all second derivatives $\ddot{G}_{Bij}$ appearing in P_N can be removed, so that P_N gets replaced by another polynomial Q_N depending solely on the $\dot{G}_{Bij}$'s,

$$P_N(\dot{G}_{Bij}, \ddot{G}_{Bij})\, \mathrm{e}^{\frac{1}{2}\sum G_{Bij} k_i \cdot k_j} \quad \stackrel{\mathrm{part.int.}}{\longrightarrow} \quad Q_N(\dot{G}_{Bij})\, \mathrm{e}^{\frac{1}{2}\sum G_{Bij} k_i \cdot k_j} \tag{2.8}$$

Then the integrand for the spinor loop case can be obtained by simultaneously replacing every closed cycle $\dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ki_1}$ appearing in Q_N by

$$\dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ki_1} - G_{Fi_1i_2}G_{Fi_2i_3}\cdots G_{Fi_ki_1} \tag{2.9}$$

An additional background field $\bar{A}^\mu(x)$ with constant field strength tensor $\bar{F}_{\mu\nu}$ can be simply taken into account by appropriate changes of the worldline propagators and the path integral determinant. Those are [12,19,20,21] (deleting the "bar")

$$\begin{aligned}
G_{B12} &\to \mathcal{G}_{B12} \equiv \frac{T}{2(\mathcal{Z})^2}\biggl(\frac{\mathcal{Z}}{\sin(\mathcal{Z})}\mathrm{e}^{-i\mathcal{Z}\dot{G}_{B12}} + i\mathcal{Z}\dot{G}_{B12} - 1\biggr)\\
G_{F12} &\to \mathcal{G}_{F12} = G_{F12}\frac{\mathrm{e}^{-i\mathcal{Z}\dot{G}_{B12}}}{\cos(\mathcal{Z})}\\
(4\pi T)^{-\frac{D}{2}} &\to (4\pi T)^{-\frac{D}{2}}\mathrm{det}^{-\frac{1}{2}}\biggl[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\biggr] \qquad (\mathrm{Scalar}\quad \mathrm{QED})\\
(4\pi T)^{-\frac{D}{2}} &\to (4\pi T)^{-\frac{D}{2}}\mathrm{det}^{-\frac{1}{2}}\biggl[\frac{\tan(\mathcal{Z})}{\mathcal{Z}}\biggr] \qquad (\mathrm{Spinor}\quad \mathrm{QED})
\end{aligned} \tag{2.10}$$

These expressions should be understood as power series in the matrix $\mathcal{Z}^{\mu\nu} \equiv eTF^{\mu\nu}$.

Thus one obtains the following generalization of (2.6), representing the scalar QED N - photon scattering amplitude in a constant field [19,20]:

$$\begin{aligned}
\Gamma_{\mathrm{scal}}[\{k_i,\varepsilon_i\}] &= (-ie)^N (2\pi)^D \delta(\sum k_i)\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}} e^{-m^2T}\mathrm{det}^{-\frac{1}{2}}\biggl[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\biggr]\\
&\times \prod_{i=1}^{N}\int_0^T d\tau_i \exp\biggl\{\sum_{i,j=1}^{N}\biggl[\frac{1}{2}k_i\cdot\mathcal{G}_{Bij}\cdot k_j - i\varepsilon_i\cdot\dot{\mathcal{G}}_{Bij}\cdot k_j + \frac{1}{2}\varepsilon_i\cdot\ddot{\mathcal{G}}_{Bij}\cdot\varepsilon_j\biggr]\biggr\}\,|_{\mathrm{multi-linear}}
\end{aligned} \tag{2.11}$$

The cycle replacement rule (2.9) can also be generalized to the constant field case.

The master formula (2.11) is valid off-shell, and can therefore be used to construct the quenched (one-electron-loop) Euler-Heisenberg Lagrangians in scalar or spinor QED at the n - loop order by starting from the one-loop $2(n-1)$ photon amplitude in a constant field, and sewing off pairs of legs. (Alternatively, one can use the worldline formalism also to calculate the Euler-Heisenberg Lagrangians directly at the multiloop level [20,22,13].)

3. The one-loop Euler-Heisenberg Lagrangians

The one-loop EHL's involve only the determinant factors in (2.10). After renormalization, one has

$$\begin{aligned}
\mathcal{L}^{(1)}_{\rm scal}(F) &= \frac{1}{16\pi^2}\int_0^\infty \frac{dT}{T^3}\,{\rm e}^{-m^2T}\left[\frac{(eaT)(ebT)}{\sinh(eaT)\sin(ebT)} + \frac{e^2}{6}(a^2-b^2)T^2 - 1\right]\\
\mathcal{L}^{(1)}_{\rm spin}(F) &= -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\,{\rm e}^{-m^2T}\left[\frac{(eaT)(ebT)}{\tanh(eaT)\tan(ebT)} - \frac{e^2}{3}(a^2-b^2)T^2 - 1\right]
\end{aligned} \tag{3.1}$$

Here a,b are the two invariants of the Maxwell field, related to $\mathbf{E}$, $\mathbf{B}$ by $a^2-b^2 = B^2-E^2, \quad ab = \mathbf{E}\cdot\mathbf{B}$. The subtraction terms in the square brackets implement the renormalization of charge and vacuum energy. The subscripts distinguish between Scalar and Spinor QED, the superscripts refer to the loop order.

The EHL's contain the information on the N photon amplitudes in the low energy limit where all photon energies are small compared to the electron mass, $\omega_i \ll m$. At the one-loop four photon level, this construction of the low-energy amplitude from the effective Lagrangian is a textbook exercise (see, e.g., 23). Even the one-loop (on-shell) N - photon amplitudes in this limit can still be written quite concisely using spinor helicity techniques [24].

Except for the purely magnetic field case, the parameter integrals in (3.1) contain poles, leading to an imaginary part of the EHL's. A simple application of the residue theorem gives Schwinger's famous formulas [25],

$$\begin{aligned}
{\rm Im}\mathcal{L}^{(1)}_{\rm spin}(E) &= \frac{m^4}{8\pi^3}\beta^2\sum_{k=1}^\infty \frac{1}{k^2}\,\exp\left[-\frac{\pi k}{\beta}\right]\\
{\rm Im}\mathcal{L}^{(1)}_{\rm scal}(E) &= \frac{m^4}{16\pi^3}\beta^2\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k^2}\,\exp\left[-\frac{\pi k}{\beta}\right]
\end{aligned} \tag{3.2}$$

($\beta = eE/m^2$). The kth term in these sums is interpreted as representing the instability of the vacuum with respect to the coherent production of k electron-positron (reps. scalar-antiscalar) pairs by the field (vacuum tunneling). In the following we will concentrate on the weak field limit $\beta \ll 1$ where only the $k=1$ term is relevant.

4. The two-loop Euler-Heisenberg Lagrangian

The two-loop EHL, involving one internal photon exchange in the loop, was first calculated by V.I. Ritus, both for Spinor [4] and Scalar QED [26]. This resulted in a type of rather intractable two-parameter integrals, on which also later recalculations were not able to substantially improve [27,20,22]. However, the first few coefficients of the weak-field expansions of the two-loop EHL's have been computed [20,28,29], and there are simple closed-form expressions for the case of a (euclidean) self-dual field [30]. Those allow one to extend the one-loop calculation of the on-shell low energy N photon amplitudes, mentioned above, to the two-loop level for the case where all the photon helicities are the same [30].

As to the imaginary parts, the Schwinger formulas (3.2) generalize to the two-loop level as follows [31]:

$$\begin{aligned}
\mathrm{Im}\mathcal{L}^{(2)}_{\mathrm{spin}}(E) &= \frac{m^4}{8\pi^3}\beta^2\sum_{k=1}^{\infty}\alpha\pi K_k^{\mathrm{spin}}(\beta)\,\exp\left[-\frac{\pi k}{\beta}\right]\\
\mathrm{Im}\mathcal{L}^{(2)}_{\mathrm{scal}}(E) &= \frac{m^4}{16\pi^3}\beta^2\sum_{k=1}^{\infty}(-1)^{k+1}\alpha\pi K_k^{\mathrm{scal}}(\beta)\,\exp\left[-\frac{\pi k}{\beta}\right]
\end{aligned}\tag{4.1}$$

($\alpha=\frac{e^2}{4\pi}$) where

$$\begin{aligned}
K_k^{\mathrm{scal,spin}}(\beta) &= -\frac{c_k}{\sqrt{\beta}}+1+\mathrm{O}(\sqrt{\beta})\\
c_1 = 0, \qquad c_k &= \frac{1}{2\sqrt{k}}\sum_{l=1}^{k-1}\frac{1}{\sqrt{l(k-l)}}, \qquad k\geq 2
\end{aligned}\tag{4.2}$$

Thus at two-loop the kth Schwinger-exponential appears with a prefactor which is still a function of the field strength, and of which presently only the lowest order terms in the weak-field expansion are known. Still, things become very simple at leading order in this expansion: Adding the one-loop and two-loop EHL's, one finds, e.g. for the spinor QED case [31],

$$\mathrm{Im}\mathcal{L}^{(1)}_{\mathrm{spin}}(E)+\mathrm{Im}\mathcal{L}^{(2)}_{\mathrm{spin}}(E) \overset{\beta\to 0}{\sim} \frac{m^4\beta^2}{8\pi^3}\bigl(1+\alpha\pi\bigr)\,\mathrm{e}^{-\frac{\pi}{\beta}}\tag{4.3}$$

and this result is spin-independent (but for the normalization). In 31 it was further noted that, if one assumes that in this weak-field approximation higher order corrections just lead to an exponentiation,

$$\sum_{l=1}^{\infty}\mathrm{Im}\mathcal{L}^{(l)}(E) \overset{\beta\to 0}{\sim} \mathrm{Im}\mathcal{L}^{(1)}(E)\,\mathrm{e}^{\alpha\pi}\tag{4.4}$$

then the result can, in the tunneling picture, be related to the fact that a created pair gets born at a finite distance, and thus with a negative Coulomb binding energy.

5. An all-loop conjecture from worldline instantons

Already in earlier work (but unknown to the authors of 31 at the time) Affleck et al. [14] had invented the concept of worldline instantons, and their principal application was precisely to demonstrate the exponentiation property (4.4), although not for spinor QED but for the scalar QED case. Even though neither derivation of (4.4) (to be called "AAM conjecture" in the following) can be considered rigorous, the fact that it was obtained by two very different lines of reasoning makes us confident about its correctness. Assuming this to be the case, (4.4) is a highly remarkable result, since, despite of the simplicity of the derivation by 14 in terms of a single semi-classical instanton trajectory, it is a true all-loop result, receiving contributions from an infinite set of Feynman diagrams (although it is important for the following that only quenched (one fermion-loop) diagrams contribute in this limit). Moreover, 14 argue that (4.4) is written in terms of the physically renormalized mass, that is, the counter diagrams for mass renormalization, which normally are necessary starting from the two-loop level, have been taken care of implicitly. This again cannot be considered as rigorously proven, however what can be shown easily is that, *if* (4.4) holds, then the mass appearing in it must be the physically renormalized one. Namely, using a Borel dispersion relation one can show [28] the following general relation between the prefactor of the first Schwinger exponential and the leading asymptotic growth of the weak field expansion coefficients at fixed loop order l: Defining these coefficients by

$$\mathcal{L}^{(l)}(E) = \sum_{n=2}^{\infty} c^{(l)}(n)\Bigl(\frac{eE}{m^2}\Bigr)^{2n} \tag{5.1}$$

one has

$$\begin{aligned} c^{(l)}(n) &\overset{n\to\infty}{\sim} c_\infty^{(l)}\Gamma[2n-2] \\ \mathrm{Im}\mathcal{L}^{(l)}(E) &\sim c_\infty^{(l)}\,\mathrm{e}^{-\frac{\pi m^2}{eE}} \end{aligned} \tag{5.2}$$

with constants $c_\infty^{(l)}$. This implies, in particular, that the leading factorial growth order of the expansion coefficients must be the same at each loop order, and in 28 it has been shown that already at the two-loop level this holds true if and only if the renormalized mass is the on-shell one.

6. Relation to the multiloop N photon amplitudes at large N

Now, the AAM conjecture is remarkable not only for its simplicity, but also for the fact that, despite of arising from a true all-order loop summation, the result is

analytic in α. This appears to run contrary to many arguments which have been given, starting with Dyson's classic 1952 paper [32], to show that S-Matrix elements in QED can never be analytic in α. Now the Schwinger pair creation rate is not itself an S-matrix element, but it can, at any loop order, be related to the N-photon amplitudes at large N using the above relations (5.1),(5.2) and the already mentioned standard procedure for converting the weak field expansion coefficients into low-energy photon amplitudes (this involves also an extension of the AAM conjecture from the electric field case to the general constant field or at least self-dual field case [30,33], as well as other modest assumptions). This led one of the authors and G.V. Dunne to conjecture in 2004 [34] that the perturbation series for the N photon amplitudes, albeit divergent for the full amplitude, is indeed convergent at the level of the quenched approximation. This conjecture is not in contradiction with existing general theorems on the QED perturbation series, and extends a corresponding conjecture made by Cvitanovic in 1977 [35] for the $g-2$ factor. If true, it would indicate extensive cancellations between Feynman diagrams, presumably due to gauge invariance.

Referring for the details to [34,30,33,17], let us just state here that this line of reasoning leads to a number of nontrivial predictions starting from the three-loop level. Namely, we expect to find the expansion coefficients of the three-loop EHL (for both Scalar and Spinor QED) to have the following three properties:

(1) $\lim_{n\to\infty} \frac{c^{(3)}(n)}{c^{(1)}(n)} = \frac{1}{2}\alpha^2$.
(2) Only the quenched part of the EHL should contribute to this limit.
(3) The convergence of $\frac{c^{(3)}(n)}{c^{(1)}(n)}$ to $\frac{c^{(3)}_\infty}{c^{(1)}_\infty}$ (from the first eq. in (5.2)) should not be slower than the one for the corresponding two-loop to one-loop ratio.

Unfortunately, a calculation of the three-loop EHL has so far been proven technically out of reach. However, motivated by work of Dunne and Krasnansky [36,37] on EHL's in various space-time dimensions it was shown in 17 that the whole above machinery can, *mutatis mutandis*, be transferred to the computationally simpler context of QED in 1+1 dimensions. In particular, all of the above three-loop predictions can be generalized to this case, changing only the definition of the ratios,

$$\frac{c^{(l)}(n)}{c^{(1)}(n)} \to \frac{c^{(l)}_{2D}(n)}{c^{(1)}_{2D}(n+l-1)} \tag{6.1}$$

and changing α to $2\frac{e^2\pi}{m^2}$. Preliminary results on a calculation of the three-loop EHL in 2D Spinor QED using the Feynman diagram approach were presented at the QFEXT09 conference [38], however this approach ultimately failed, since it led to parameter integrals with spurious IR divergences. Here, we will present the results of a new run on the calculation of the same three-loop EHL using the worldline formalism [39], which has allowed us to obtain this Lagrangian in terms of manifestly (IR and UV) finite parameter integrals.

7. The three-loop Euler-Heisenberg Lagrangian for 2D QED

At the three-loop level, there are three Feynman diagrams contributing to the EHL in spinor QED, shown in fig. 2:

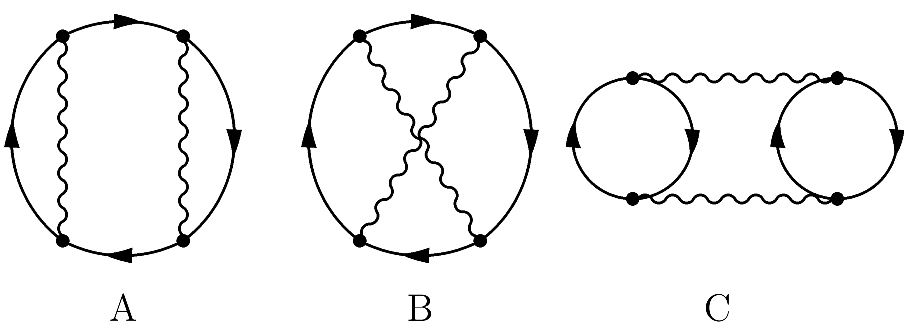

Fig. 2. Feynman diagram representation of the three-loop EHL.

Here the solid line stands for the fermion propagator in the constant field with field strength tensor $F = \left(\begin{smallmatrix} 0 & f \\ -f & 0 \end{smallmatrix}\right)$. The calculation of the non-quenched diagram C is straightforward, and leads to a fourfold proper-time integral

$$\mathcal{L}^{3C}(f) = \frac{e^3}{16\pi^3 f}\int_0^\infty dz dz' d\hat{z} dz'' \frac{\sinh z \sinh z' \sinh \hat{z} \sinh z''}{[\sinh(z+z')\sinh(\hat{z}+z'')]^2} \\ \times \frac{\mathrm{e}^{-2\kappa(z+z'+\hat{z}+z'')}}{\sinh z \sinh z' \sinh(\hat{z}+z'') + \sinh \hat{z} \sinh z'' \sinh(z+z')} \tag{7.1}$$

where $\kappa = m^2/2ef$. From this we have obtained, by numerical integration using MATHEMATICA, the first 12 weak field expansion coefficients, which was sufficient to verify that they are indeed asymptotically suppressed, even exponentially, with respect to the asymptotic prediction of the (2D analogue of) the AAM formula. Thus point (2) of our three-loop predictions above has been settled.

To the contrary, the parameter integral representations which we have obtained for the quenched diagrams A and B are lengthy, and cannot be given here in full. Their structure is

$$\mathcal{L}^{3(A+B)}(f) = -\frac{e^4}{(4\pi)^3}\int_0^\infty \frac{dT}{T^2}\mathrm{e}^{-m^2 T}\frac{Z}{\tanh Z}\prod_{i=1}^4 \int_0^T d\tau_i \\ \times \Big(2I_{1234} + I_{1324} + 4I_{123} + 2I_{12} + 4I_{13} + I_{12,34} + 2I_{13,24}\Big) \tag{7.2}$$

where $Z = efT$ and, for example,

$$I_{ijkl} = \frac{\mathrm{tr}\big(\{ijkl\}_S\big)}{\Delta} \tag{7.3}$$

with (compare (2.9))

$$\{i_1 i_2 \dots i_n\}_S := \hat{\dot{\mathcal{G}}}_{Bi_1i_2}\hat{\dot{\mathcal{G}}}_{Bi_2i_3}\cdots\hat{\dot{\mathcal{G}}}_{Bi_ni_1} - \mathcal{G}_{Fi_1i_2}\mathcal{G}_{Fi_2i_3}\cdots\mathcal{G}_{Fi_ni_1} \tag{7.4}$$

Here we have further introduced $\hat{\dot{\mathcal{G}}}_{Bij} := \dot{\mathcal{G}}_{Bij} - \dot{\mathcal{G}}_{Bii} + \mathcal{G}_{Fii}$, and Δ is a determinant also involving the worldline Green's functions. Note that (7.2) represents the sum of both diagrams A and B.

8. Conclusions

We have summarized here what is presently known about multiloop corrections to the EHL, concentrating on the potential of such corrections to yield information on the high-order behavior of the perturbation series in QED. As part of a long-term effort to prove or disprove "quenched convergence" for the case of the photon S-matrix, we have presented a parameter integral representation of the three-loop EHL in 2D QED. Although the extraction of the weak-field expansion coefficients from this representation and verification of the remaining predictions implied by this conjecture at the three-loop level (points (1) and (3) above) will still require very substantial work, we are confident that we will have definite results to show by the time of QFEXT13!

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